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MATHEMATICAL DESCRIPTIONS OF ANISOTROPIC FRICTION

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Abstract—A linear model of anisotropic dry friction is completed with properties about principal directions of friction and friction symmetries. Next, a nonlinear model of frictional anisotropy is formulated. It describes the anisotropy with arbitrary number of principal directions of friction and with arbitrary shape of a friction force hodograph. Types of frictional anisotropy are classified with the aid of symmetry groups. Figures illustrate different frictional anisotropy classes and the influence of anisotropic friction on a material point motion.

1. INTRODUCTION

Friction which depends on the direction of sliding is called anisotropic friction. A deviation in the friction force from the direction of sliding and a dependence of the friction magnitude on the sliding direction are typical features of contacts with frictional anisotropy. There are exceptions to this rule. For the isotropic friction case and for particular sliding directions which are called the principal directions of friction, the friction force is opposite to the sliding direction.

Friction depends on the sliding direction as a result of anisotropic surface roughness originated by machining and/or wear. Wear processes cause progressive changes in the initial surface roughness and in the frictional anisotropy. The topography of a surface has a strong influence on dry friction and mixed friction. It has been investigated experimentally by Diacenko (1946), Rabinowicz (1957), Sharpin (1957), Halaunbrenner (1960) and Zieliński (1964). The influence of directional surface structure on friction decreases as surface smoothness increases but it does not vanish completely. At present, there are some possibilities of controlling the machining process and a prescribed surface roughness can be produced.

Rabinowicz (1957) and Halaunbrenner (1960) realized experimental investigations of the friction force component normal to the sliding direction. This component occurs when the friction force direction is different from the motion direction. Sharpin (1957) reported that the type of anisotropy and magnitude of friction between two surfaces with known roughnesses and similar hardnesses depend on the relative position of the surfaces. Various roughness magnitudes have been taken into consideration. The root mean square of the surface roughness had a value ranging from 0.25 to a few micrometres. In an extreme case it reached 42 μm (Halaunbrenner, 1960).

Mechanical properties of materials depending on a direction are another reason for frictional anisotropy. The anisotropy of mechanical properties appears essentially in crystalline and fibrous materials. It was observed that the friction depends on the orientation of the molecular chains of PTFE and the sliding direction (Tabor and Williams, 1961). Friction anisotropy has been obtained in testing unidirectionally crystallized aluminium–copper alloys (Topfenec *et al.*, 1984). The friction of composites depends on fiber orientation with respect to the sliding direction as was experimentally demonstrated by Roberts (1984) and Minford and Prewo (1985).

Measurements of the effect of sliding orientation on friction between rough surfaces or composites show that it is a well defined phenomenon. They show that the friction magnitude may change by up to 30% depending on the orientation (Minford and Prewo, 1985), whereas its direction may differ from the sliding direction by an angle equal to a few degrees (Rabinowicz, 1957; Halaunbrenner, 1960). The friction magnitude and the angle of inclination of the friction force are continuous functions of the sliding direction angle.

Anisotropy of mechanical, thermal, electrical, magnetic and optical properties is a fundamental feature of crystals (Nye, 1957). Surface properties of crystals such as: friction (Seal, 1957; Bailey and Gwathmey, 1962; Bowden and Brooks, 1966; Buckley, 1968; Steijn, 1969; Casey and Wilks, 1973; Enomoto and Tabor, 1980; Ohmae, 1980; Miyoshi and Buckley, 1982), wear (Duwell, 1969), hardness (Brooks *et al.*, 1971), surface tension, adsorption, chemical activity, etc. are also anisotropic. The frictional anisotropy has been measured for single crystals of diamond (Seal, 1957; Bowden and Brooks, 1966; Casey and Wilks, 1973; Enomoto and Tabor, 1980), copper (Bailey and Gwathmey, 1962; Ohmae, 1980), rutile (Steijn, 1969), SiC (Miyoshi and Buckley, 1982), magnesium oxide (Bowden and Brooks, 1966) and lithium fluoride (Bowden and Brooks, 1966). Furthermore, investigations of the frictional anisotropy for cobalt, beryllium, rhenium, titanium, aluminium, iron and sapphire are summarized by Buckley (1968) and Ohmae (1980).

Diamond natural faces are usually extremely rough. For precision optical works they are polished. Since the Middle Ages it has been known that diamond offers different resistances to polishing in different directions on different surfaces. It was discovered that on the same plane, there are both easier and harder polishing directions.

The frictional anisotropy of crystals results from directional differences in the arrangement of atoms in a crystal lattice. Investigations show that friction and wear depend on crystallographic plane and on the sliding direction orientation with respect to the crystallographic system. The frictional anisotropy of single crystals achieves high values. The magnitude of change of the friction force with the sliding direction may reach 300% (Bowden and Brooks, 1966). It was found (Seal, 1957), that the frictional anisotropy magnitude depends on the relative orientation of the faces of both contacting crystals.

Orthotropic dry friction (i.e. the case in which two principal directions are orthogonal) has been analysed in a limited number of papers. In each case it was assumed *a priori* that the friction forces could be described with respect to the principal directions. Moszyński (1951), Ziemba (1952) and Vantorin (1962) assumed that the extremity of the orthotropic friction force vector draws different curves on a plane and they carried out an analysis on this basis. Huber (1957) considered the orthotropic friction to be analogous to a distribution of normal stresses in a plane state of stress. Fredriksson (1976), Michałowski and Mróz (1978), Ziegler (1981), Curnier (1984) and others assumed that the anisotropic friction forces can be derived in a similar way as constitutive relations in the theory of plasticity.

In this paper a phenomenological model of frictional anisotropy formulated by Zmitrowicz (1977, 1978 and 1981) is extended into a nonlinear model and is completed with new properties, interpretations and graphical illustrations. Tensor algebra in connection with symmetry groups provides a good mathematical apparatus for describing anisotropic friction. We shall apply symmetry group theory to obtain a mathematical classification of anisotropies.

2. A LINEAR MODEL OF ANISOTROPIC FRICTION

According to the thermodynamical theory of constitutive equations of friction (Zmitrowicz, 1987, 1988), the friction force vector $\mathbf{t}$ can be a function of the slip velocity unit vector $\mathbf{v}$ and the normal pressure N, e.g.

$$\mathbf{t} = -Nf(\mathbf{v}) \tag{1}$$

where

$$\mathbf{t} = t^i \mathbf{k}_i \in \underset{1}{\mathscr{E}_2} \quad i = 1, 2 \tag{2}$$

$$\mathbf{v} = v^j \mathbf{e}_j \in \underset{2}{\mathscr{E}_2} \quad j = 1, 2 \tag{3}$$

$$|\mathbf{v}| = 1 \tag{4}$$

$$N \in \mathscr{R}^+. \tag{5}$$

$\underset{1}{\mathscr{E}}_2$ and $\underset{2}{\mathscr{E}}_2$ are two two-dimensional vector spaces and $\{\mathbf{k}_i\}$ is an orthogonal basis of unit vectors in $\underset{1}{\mathscr{E}}_2$ and $\{\mathbf{e}_j\}$ is an arbitrary unit vector basis in $\underset{2}{\mathscr{E}}_2$. Let us consider a linear case of eqn (1), i.e.

$$\mathbf{t} = -N\mathbf{C}_1\mathbf{v} \tag{6}$$

where $\mathbf{C}_1$ is a nonsingular, second-order tensor. This tensor belongs to a space $\mathscr{T}_2$ being the tensor product of the vector spaces $\underset{1}{\mathscr{E}}_2$ and $\underset{2}{\mathscr{E}}_2$

$$\mathscr{T}_2 = \underset{1}{\mathscr{E}}_2 \otimes \underset{2}{\mathscr{E}}_2. \tag{7}$$

The tensor $\mathbf{C}_1$ can be expressed as a linear combination of the four tensor basis elements $\mathbf{k}_i \otimes \mathbf{e}_j$

$$\mathbf{C}_1 = C^{ij}\mathbf{k}_i \otimes \mathbf{e}_j \in \mathscr{T}_2. \tag{8}$$

The tensor $\mathbf{C}_1$ was called Coulomb friction tensor, Zmitrowicz (1977, 1978, 1981). Equation (6) has the following form in the representation notation:

$$\mathbf{t} = -N(C^{ij}\mathbf{k}_i \otimes \mathbf{e}_j)(v^l\mathbf{e}_l) = -NC^{ij}v_j\mathbf{k}_i. \tag{9}$$

Generally the vectors $\mathbf{t}$ and $\mathbf{v}$ are not colinear. They have different norms and directions. The mapping $\mathbf{t}$ from a unit sphere in $\underset{2}{\mathscr{E}}_2$ into $\underset{1}{\mathscr{E}}_2$ is not linear but it can be extended into a linear mapping from $\underset{2}{\mathscr{E}}_2$ into $\underset{1}{\mathscr{E}}_2$.

Some physical properties of nature are described by second order tensors, e.g. stress, heat conduction, diffusion, electric conduction, magnetic permeability, etc. Here, the second-order friction tensor characterizes a linear model of frictional anisotropy. Equation (6) is a generalization of the models of dry friction formulated by Amontons and Coulomb.

The four coefficients defining the representation of the tensor $\mathbf{C}_1$ with respect to a given tensor basis will be obtained from the friction force distribution over the contact. This approach provides a physical interpretation of the friction tensor coefficients. We shall specify the class of friction problems which can be described by this linear model.

Two arbitrary directions 0ξ and 0η called experimental directions at the contact area are defined with respect to the orthogonal reference system $0xy$ by the angles ε_x and ε_y (Fig. 1). During the sliding, the friction forces $t_\xi = \mu_\xi N$ and $t_\eta = \mu_\eta N$ acting on a line segment of

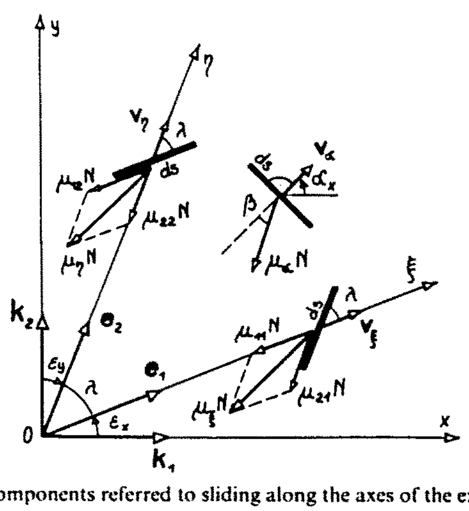

Fig. 1. Friction force components referred to sliding along the axes of the experimental system $0\xi\eta$.

length ds inclined at an angle λ are oriented along the directions 0ξ and 0η. Slip occurs in the positive sense of the experimental directions. The four parameters μ_{ij} $(i, j = 1, 2)$, which are called the coefficients of friction, are defined by decomposition of the friction forces into components in the directions 0ξ and 0η. The sign convention for the friction force components is shown in Fig. 1.

Hypothesis 1. Let us assume that the friction force acting on the sliding segment ds inclined at angle α_x with respect to the $0x$ axis can be defined as the sum of two friction force components. These components can be interpreted as the friction forces acting on ds projections when they are sliding along the directions 0ξ and 0η.

With the above assumption, the friction forces are described by formulae analogous to the relations used in classical elasticity to define the stresses acting on a surface element. The sides of the triangular element are parallel to the axes of the coordinate system $0\xi\eta$ (Fig. 2). The stress p_α acting on the side ds is equal to the sum of the stresses acting on the sides ds_ξ and ds_η. These stresses are decomposed into components parallel to the directions 0ξ and 0η. Since we are not using conditions in which equilibrium of the moments holds, all relations between the stresses are referred to an asymmetric state of stress. The assumed analogy between the stresses in a plane state of tension and friction forces gives us a correspondence between the stress components σ_ξ, $\tau_{\eta\xi}$, $\tau_{\xi\eta}$, σ_η and the friction coefficients μ_{11}, μ_{12}, μ_{21}, μ_{22}.

Substituting the friction coefficients resulting from the assumed analogy into the stress magnitude expression

$$p_\alpha = [(\sigma_\alpha)^2 + (\tau_\alpha)^2]^{1/2} \tag{10}$$

we obtain the anisotropic friction coefficient for any sliding direction α_x. It can be given by

$$\mu_\alpha = [(\mu_\alpha^\parallel)^2 + (\mu_\alpha^\perp)^2]^{1/2} \tag{11}$$

where $\mu_\alpha^\parallel$ and $\mu_\alpha^\perp$ are friction coefficients of the friction force component colinear with the sliding direction and normal to this direction. The coefficients $\mu_\alpha^\parallel$ and $\mu_\alpha^\perp$ are determined by the stress relations for σ_α and τ_α to give

$$\mu_\alpha^\parallel = \frac{1}{\cos(\varepsilon_x + \varepsilon_y)} [\mu_{11} \cos^2(\alpha_x - \varepsilon_x) + \mu_{22} \cos^2(\alpha_y - \varepsilon_y) + (\mu_{12} + \mu_{21}) \cos(\alpha_x - \varepsilon_x) \cos(\alpha_y - \varepsilon_y)] \tag{12}$$

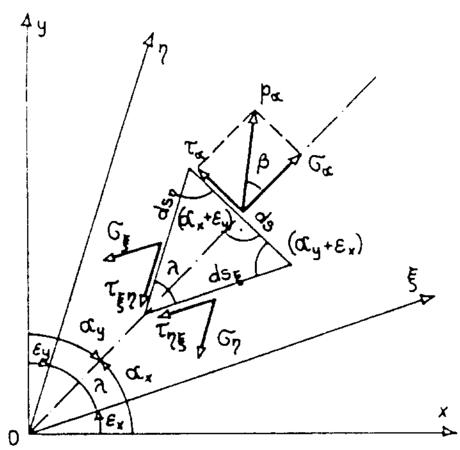

Fig. 2. Analogy between the friction force distribution at the contact and the plane state of stress.

$$\mu_x^{\perp} = \frac{1}{\cos(\varepsilon_x+\varepsilon_y)}\left[\tfrac{1}{2}\mu_{22}\sin(2\alpha_y-2\varepsilon_y)-\tfrac{1}{2}\mu_{11}(2\alpha_x-2\varepsilon_x)\right.$$
$$\left.+\mu_{21}\sin(\alpha_y-\varepsilon_y)\cos(\alpha_x-\varepsilon_x)-\mu_{12}\sin(\alpha_x-\varepsilon_x)\cos(\alpha_y-\varepsilon_y)\right]. \quad (13)$$

The friction force is inclined at an angle β to the direction of sliding. This angle is equal to the angle between the stress p_x and the sliding direction

$$\tan\beta = \frac{\mu_x^{\perp}}{\mu_x^{\parallel}}. \quad (14)$$

Substitution of double-angle identities into eqns (12), (13) and

$$\alpha_y = \frac{\pi}{2}-\alpha_x \quad (15)$$

produces $\mu_x^{\parallel}$ and $\mu_x^{\perp}$ as functions of the direction angle α_x. It is possible to find the directions of extreme $\mu_x^{\parallel}$ and $\mu_x^{\perp}$ by differentiating these functions with respect to α_x and setting the results equal to zero, as follows

$$\frac{d\mu_x^{\parallel}}{d\alpha_x} = 0 \quad (16)$$

$$\frac{d\mu_\alpha^{\perp}}{d\alpha_x} = 0. \quad (17)$$

An analysis of the relations (16), (17) shows that the directions of extreme values for both $\mu_x^{\parallel}$ and $\mu_x^{\perp}$ are orthogonal in every case. The directions of $\mu_x^{\perp}$ extreme values are inclined at an angle $\pi/4$ with respect to these of $\mu_x^{\parallel}$.

The quantities μ_x and β make it possible to calculate the friction force components and the components of the friction tensor $\mathbf{C}_1$ for any sliding direction α_x. Thus, we get the following matrix as a representation of $\mathbf{C}_1$

$$[C^{ij}] = \begin{bmatrix} \dfrac{\mu_{11}\cos\varepsilon_x+\mu_{21}\sin\varepsilon_y}{\cos(\varepsilon_x+\varepsilon_y)} & \dfrac{\mu_{12}\cos\varepsilon_x+\mu_{22}\sin\varepsilon_y}{\cos(\varepsilon_x+\varepsilon_y)} \\ \dfrac{\mu_{21}\cos\varepsilon_y+\mu_{11}\sin\varepsilon_x}{\cos(\varepsilon_x+\varepsilon_y)} & \dfrac{\mu_{22}\cos\varepsilon_y+\mu_{12}\sin\varepsilon_x}{\cos(\varepsilon_x+\varepsilon_y)} \end{bmatrix} \quad (18)$$

$$\mu_{ij}\in\mathcal{R} \quad (19)$$

$$\cos(\varepsilon_x+\varepsilon_y)\neq 0. \quad (20)$$

Condition (20) means that two different measurements of the friction coefficients along the same direction of sliding are not sufficient for the identification of friction. The contravariant components v_j of the sliding velocity unit vector with respect to the basis $\{\mathbf{e}_j\}$ are defined as

$$[v_j]^T = [\cos(\alpha_x-\varepsilon_x),\quad \cos(\alpha_y-\varepsilon_y)]^T. \quad (21)$$

The friction coefficients μ_{ij} can be identified in different practical ways. For example, we measure the friction force vectors for two arbitrary sliding directions. Next, by solving the system of four equations (9), we obtain the coefficients μ_{ij}.

The following properties of the linear model of frictional anisotropy are described: the decomposition of the second-order friction tensor, the gyroscopic part of the friction force component which refers to the antisymmetric part of the friction tensor, the eigenvalues and eigenvectors of the orthotropic and isotropic friction tensors, positive definiteness of the second-order friction tensor (Zmitrowicz, 1977, 1978, 1981). Here, they are completed by new properties and interpretations.

Property 1. The linear model of frictional anisotropy determines the friction with two, one or zero principal directions.

The friction force vector is colinear with the unit vector of slip velocity for sliding in principal directions of friction. Thus, the friction force vector expressed by (9) is related to the unit vector $\mathbf{v}$ indicating the direction in the contact by

$$-N\mathbf{C}_{\mathrm{I}}\mathbf{v} = -N\mu\mathbf{v}. \tag{22}$$

Simultaneously the component of the friction force normal to the principal direction is equal to zero. The equation

$$(\mathbf{C}_{\mathrm{I}} - \mu\mathbf{1})\mathbf{v} = \mathbf{0} \tag{23}$$

has a nontrivial solution if and only if the determinant of the coefficient matrix is equal to zero, i.e.

$$\det(\mathbf{C}_{\mathrm{I}} - \mu\mathbf{1}) = 0. \tag{24}$$

The discriminant Δ of eqn (24) determines the number of real eigenvalues of the friction tensor $\mathbf{C}_{\mathrm{I}}$. It has two real eigenvalues and two eigenvectors when

$$\Delta = (C^{11} - C^{22})^2 + 4C^{12}C^{21} > 0. \tag{25}$$

It has only one real eigenvalue and one eigenvector when $\Delta = 0$. It has no real eigenvalues and no eigenvectors when $\Delta < 0$.

The eigenvectors $\mathbf{m}_i$ of the tensor $\mathbf{C}_{\mathrm{I}}$ referring to the eigenvalues μ_i satisfy the equation

$$\mathbf{C}_{\mathrm{I}}\mathbf{m}_i = \mu_i\mathbf{m}_i. \tag{26}$$

We can employ the friction tensor representation, determined with respect to the tensor basis $\mathbf{k}_i \otimes \mathbf{k}_l$,

$$\mathbf{C}_{\mathrm{I}} = C^{ij}A^l_j\mathbf{k}_i \otimes \mathbf{k}_l \quad i, j, l = 1, 2 \tag{27}$$

where

$$\mathbf{e}_j \cdot \mathbf{k}^l = A^l_j = \begin{bmatrix} \cos\varepsilon_x & \sin\varepsilon_x \\ \sin\varepsilon_y & \cos\varepsilon_y \end{bmatrix}. \tag{28}$$

Then, the unit tensor in eqn (23) has the form

$$\mathbf{1} = \delta^{il}\mathbf{k}_i \otimes \mathbf{k}_l \tag{29}$$

where δ^{il} is the Kronecker delta. If the eigenvectors are wanted in the basis of the orthogonal system $0xy$, they can be determined from (26) using (27) as

$$\mathbf{m}_i = m^l_i\mathbf{k}_l. \tag{30}$$

The friction tensor $\mathbf{C}_1$ representation in the basis $\mathbf{e}_i \otimes \mathbf{e}_j$ is

$$\mathbf{C}_1 = C^{ij}(A_i^l)^{-1}\mathbf{e}_l \otimes \mathbf{e}_j \tag{31}$$

and the unit tensor

$$\mathbf{1} = \frac{1}{\cos^2(\varepsilon_x+\varepsilon_y)}\delta^{ik}A_i^l A_k^j \mathbf{e}_l \otimes \mathbf{e}_j \tag{32}$$

where

$$\mathbf{k}_l = (A_l^k)^{-1}\mathbf{e}_k \tag{33}$$

and

$$\mathbf{e}_k \cdot \mathbf{e}_l = \begin{bmatrix} 1 & \sin(\varepsilon_x+\varepsilon_y) \\ \sin(\varepsilon_x+\varepsilon_y) & 1 \end{bmatrix}. \tag{34}$$

Using (27) and (28) we get the eigenvectors associated with the experimental system $0\xi\eta$,

$$\mathbf{m}_i = m_i^l \mathbf{e}_l. \tag{35}$$

Substitution of the tensors (27) and (29) or (31) and (32) in eqn (24) gives the same eigenvalues.

The following classification of the frictional anisotropy can be established: anisotropy with two principal directions, anisotropy with one principal direction and anisotropy without principal direction. Considering the inclination angle β in the case of the anisotropy with one principal direction, we distinguish a class of permanently non-positive and permanently non-negative functions of the angle β. However, the anisotropy without principal directions has functions of the angle β being permanently positive or permanently negative.

Property 2. Anisotropic, orthotropic and isotropic friction can be distinguished with the aid of the friction tensor $\mathbf{C}_1$ and the group of symmetry.

Every mathematical description is endowed with an automorphism group called the symmetry group of this description. The symmetry group of the linear function (9) is identical with the symmetry group of the friction tensor $\mathbf{C}_1$. A set of tensors with the tensor composition operation as the group multiplication operation forms a symmetry group of the tensor. The composition of two group elements is a group element. Here, we employ rotations and mirror reflections (space symmetries) as symmetry operations. The group axioms will be satisfied, i.e. the group multiplication operation is associative, there exist an identity element and inverses of all group elements. The importance of groups lies in the fact that they may be used to describe frictional symmetries, as will be indicated.

The subgroup $\mathscr{G}_{\mathbf{C}_1}$ of the full orthogonal group $\mathscr{O}$ is a group of symmetry of the tensor $\mathbf{C}_1$ and is defined by

$$\mathscr{G}_{\mathbf{C}_1} = \left\{\mathbf{R} : \mathbf{R} \in \mathscr{O}, \left(\overset{2}{\underset{1}{\otimes}} \mathbf{R}\right) \cdot \mathbf{C}_1 = \mathbf{C}_1\right\} \tag{36}$$

where the operation of two compositions is denoted by $\overset{2}{\underset{1}{\otimes}}$ and can be written as

$$\left(\overset{2}{\underset{1}{\otimes}} \mathbf{R}\right) \cdot \mathbf{C}_1 = C^{ij}\mathbf{R}\mathbf{k}_i \otimes \mathbf{R}\mathbf{e}_j \equiv \mathbf{R}\mathbf{C}_1\mathbf{R}^T. \tag{37}$$

From (36) and (37) we get an equivalent formulation

$$\mathbf{R}\mathbf{C}_1 = \mathbf{C}_1\mathbf{R}. \tag{38}$$

The full orthogonal group $\mathscr{O}$ contains all orthogonal transformations satisfying the following

$$\mathbf{R}^{-1} = \mathbf{R}^T, \quad \det \mathbf{R} = \pm 1. \tag{39}$$

The symmetry group of orthotropic friction has the trivial subgroup $\{\pm\mathbf{1}\}$ and the subgroup of mirror reflections with respect to two-dimensional subspaces orthogonal to the principal directions of friction, i.e.

$$\mathscr{G}_{C_1} = \{\pm\mathbf{1}, \mathbf{J}_{\mathbf{m}_1}, \mathbf{J}_{\mathbf{m}_2}\}. \tag{40}$$

The description (40) denotes the fourth-order subgroup ($\mathscr{G}_{C_1} \in \mathscr{O}$) generated by $\mathbf{1}$, $-\mathbf{1}$, $\mathbf{J}_{\mathbf{m}_1}$ and $\mathbf{J}_{\mathbf{m}_2}$. $\mathbf{1}$ is an identity transformation and $-\mathbf{1}$ is called the central inversion transformation. The mirror reflection with regard to two-dimensional subspace orthogonal to the principal direction $\mathbf{m}_i$ can be obtained by a composition of the rotation about $\mathbf{m}_i$ and the central inversion

$$\mathbf{J}_{\mathbf{m}_i} = -\mathbf{1}\mathbf{R}^{\pi}_{\mathbf{m}_i} \quad i = 1, 2. \tag{41}$$

The tensor $\mathbf{R}^{\pi}_{\mathbf{m}_i}$ describes the rotation about the axis $\mathbf{m}_i$ through the angle of rotation π. The mirror reflections on two mutually perpendicular planes characterize orthotropic friction. All composition operations for elements of the orthotropic friction symmetry group may be given in the form of the following multiplication table (Cayley square).

	$\mathbf{1}$	$-\mathbf{1}$	$\mathbf{J}_{\mathbf{m}_1}$	$\mathbf{J}_{\mathbf{m}_2}$
$\mathbf{1}$	$\mathbf{1}$	$-\mathbf{1}$	$\mathbf{J}_{\mathbf{m}_1}$	$\mathbf{J}_{\mathbf{m}_2}$
$-\mathbf{1}$	$-\mathbf{1}$	$\mathbf{1}$	$\mathbf{J}_{\mathbf{m}_2}$	$\mathbf{J}_{\mathbf{m}_1}$
$\mathbf{J}_{\mathbf{m}_1}$	$\mathbf{J}_{\mathbf{m}_1}$	$\mathbf{J}_{\mathbf{m}_2}$	$\mathbf{1}$	$-\mathbf{1}$
$\mathbf{J}_{\mathbf{m}_2}$	$\mathbf{J}_{\mathbf{m}_2}$	$\mathbf{J}_{\mathbf{m}_1}$	$-\mathbf{1}$	$\mathbf{1}$

(42)

Any second-order symmetric tensor may be the tensor of orthotropic friction. It has the group of symmetry of the type (40) and satisfies the condition

$$\mathbf{C}_1 = \mathbf{C}_1^T. \tag{43}$$

The second-order tensor is symmetric if its representation matrix is symmetric. Taking the friction tensor in the form used for determining the principal directions of friction (27), we find restrictions on μ_{ij}, ε_x and ε_y. Thus, the tensor (27) is symmetric if

$$\mu_{12} = \mu_{21} \tag{44}$$

for any $\mu_{ij} \in \mathscr{R}$ and ε_x, ε_y such that (20) holds. The condition (43) determines a number of independent elements of the orthotropic friction tensor $\mathbf{C}_1$.

The second-order orthotropic friction tensor has two real eigenvalues and two mutually orthogonal eigenvectors. It is a classical result of linear algebra (Eringen, 1967).

Anisotropic friction has only a trivial two-element group of symmetry

$$\mathscr{G}_{C_1} = \{\pm\mathbf{1}\}. \tag{45}$$

There are no restrictions on the anisotropic friction tensor representation. The multiplication operations of the symmetry transformations (45) are given in the following Cayley square

$$\begin{array}{c|cc} & 1 & -1 \\ \hline 1 & 1 & -1 \\ -1 & -1 & 1 \end{array} \tag{46}$$

The multiplication tables (42) and (46) show that the symmetry groups (40) and (45) are commutative (Abelian) groups. This is manifested in the symmetry of the tables with respect to the main diagonal.

The full orthogonal group $\mathcal{O}$ is the symmetry group of isotropic friction

$$\mathcal{G}_{C_1} = \mathcal{O}. \tag{47}$$

This group is a continuous group. Any second-order spherical tensor is the isotropic friction tensor and its group of symmetry is of type (47). The spherical tensor has the following form

$$\mathbf{C}_1 = \mu\mathbf{1}, \quad \mu \in \mathcal{R}. \tag{48}$$

Taking the tensor (27) we get restrictions on μ_{ij}, ε_x and ε_y. Thus, the friction tensor $\mathbf{C}_1$ is spherical if

$$\mu_{11} = \mu_{22}, \quad \mu_{12} = \mu_{21} = 0 \tag{49}$$

$$\varepsilon_x = -\varepsilon_y \tag{50}$$

for any $\mu_{ij} \in \mathcal{R}$ and ε_x, ε_y, with restriction (20). From (48) it is seen that there is one independent element of the isotropic friction tensor $\mathbf{C}_1$.

It was proved (Zmitrowicz, 1981) that the anisotropy tensor maps a circle into an ellipse. This geometrical image of the friction force vectors is called the hodograph of the friction force. A circle is the isotropic friction force image. The ellipse drawn by the friction force vectors attached to the origin of the coordinate system $0XY$ is described by the following equation

$$aX^2 + 2bXY + cY^2 = N^2 \tag{51}$$

where

$$a = \frac{1}{e^2}[(C^{21})^2 + (C^{22})^2] \tag{52}$$

$$b = -\frac{1}{e^2}(C^{12}C^{22} + C^{11}C^{21}) \tag{53}$$

$$c = \frac{1}{e^2}[(C^{11})^2 + (C^{12})^2] \tag{54}$$

$$e = C^{11}C^{22} - C^{12}C^{21}. \tag{55}$$

Sometimes it is called the cross-section of the friction cone. With the aid of eqns (51)–(55) we formulate the static equilibrium condition between the tangential force to the contact and the friction force, i.e. adherence condition. Let

$$\mathscr{B} = \{\mathbf{f}: \mathbf{f} = (X, Y), X \in \mathscr{R}, Y \in \mathscr{R}\} \tag{56}$$

be a set of all possible tangential forces $\mathbf{f}$ acting at the contact area. Then a set

$$\mathscr{C} = \{\mathbf{f}: \mathbf{f} \in \mathscr{B}, aX^2 + 2bXY + cY^2 \leqslant N^2; \text{ for fixed } \mathbf{C}_1\} \tag{57}$$

defines the adherence condition for anisotropic friction. For isotropic friction ($\mathbf{C}_1 = \mu\mathbf{1}$) we have

$$\mathscr{C} = \{\mathbf{f}: \mathbf{f} \in \mathscr{B}, X^2 + Y^2 \leqslant \mu^2 N^2; \text{ for fixed } \mu\}. \tag{58}$$

In the case of static friction, the question of the friction force direction is simple. This force is always opposite to the tangential force and both forces have the same magnitude. The static friction force is a reaction.

Some properties of the linear model will be represented by diagrams in Appendix A.

3. A NONLINEAR MODEL OF ANISOTROPIC FRICTION

Let us consider a constitutive equation of friction force vector (Zmitrowicz, 1987, 1988) of the form (1) where the friction force vector is a polynomial function of the form

$$\mathbf{t} = -N(\mathbf{C}_1\mathbf{v} + \mathbf{C}_2\mathbf{v}^3 + \cdots + \mathbf{C}_n\mathbf{v}^{2n-1}) \tag{59}$$

where

$$\mathbf{C}_n\mathbf{v}^{2n-1} \equiv \mathbf{C}_n \cdot \underbrace{(\mathbf{v} \otimes \mathbf{v} \otimes \cdots \otimes \mathbf{v})}_{2n-1 \text{ copies}} \equiv [(\mathbf{C}_n \underbrace{\mathbf{v})\mathbf{v} \ldots]\mathbf{v}}_{2n-1 \text{ copies}}. \tag{60}$$

The operation $(\cdot)$ means the full tensor composition. The vector function (59) is a polynomial with respect to the vector argument and according to the objectivity axiom only odd order terms must be included in the polynomial. In this paper, the first two terms of the polynomial will be taken into consideration. The second-order tensor $\mathbf{C}_1$, has been defined by (8). The fourth-order tensor $\mathbf{C}_2$ belongs to the linear space $\mathscr{T}_4$ formed by the tensor product of $\underset{1}{\mathscr{E}}_2$ and three times $\underset{2}{\mathscr{E}}_2$. Sixteen tensors of the form

$$\mathbf{k}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l \quad i, j, k, l = 1, 2 \tag{61}$$

are base elements of 16-dimensional space $\mathscr{T}_4$. Then, we have

$$\mathbf{C}_2 = C^{ijkl}\mathbf{k}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l \in \mathscr{T}_4 \tag{62}$$

$$\mathscr{T}_4 = \underset{1}{\mathscr{E}}_2 \otimes \underset{2}{\mathscr{E}}_2 \otimes \underset{2}{\mathscr{E}}_2 \otimes \underset{2}{\mathscr{E}}_2 \tag{63}$$

where C^{ijkl} are arbitrary real numbers. The tnsors $\mathbf{C}_1$ and $\mathbf{C}_2$ are of even order.

The assumption of a polynomial for the function $\mathbf{t}$ is imposed by "mathematical convenience". Polynomial functions are very often used in continuum mechanics due to their simplicity and a relative completeness of achieved results. Non-polynomial representation of this function is allowed. We get the linear model of frictional anisotropy (6) if we restrict ourselves to the first term of the polynomial (59).

The total friction coefficient and the angle of friction force inclination for any sliding direction can be obtained from the following relations.

$$\mu_\alpha = N^{-1}|\mathbf{t}| \tag{64}$$

$$\cos\beta = -\frac{\mathbf{v}\cdot\mathbf{t}}{|\mathbf{t}|}. \tag{65}$$

Friction coefficients of the friction force components colinear with the sliding direction and normal to it are given by

$$\mu_\alpha^{\parallel} = N^{-1}|\mathbf{t}\cdot\mathbf{v}| \tag{66}$$

$$\mu_\alpha^{\perp} = N^{-1}|\mathbf{t}\cdot\mathbf{v}^{\perp}| \tag{67}$$

where $\mathbf{v}^{\perp}$ is a unit vector orthogonal to the sliding direction.

Let the base vectors $\{\mathbf{k}_i\}$ and $\{\mathbf{e}_j\}$ coincide with the orthogonal reference system $0xy$. The velocity unit vector components given with respect to this basis are

$$[\mathbf{v}] = [\cos\alpha_v, \sin\alpha_v]^T. \tag{68}$$

The unit vector orthogonal to the slip direction is

$$[\mathbf{v}^{\perp}] = [-\sin\alpha_v, \cos\alpha_v]^T. \tag{69}$$

Equation (59) has the following form in indicial notation

$$t^i = -N(C^{ij}v_j + C^{ilmk}v_l v_m v_k + \cdots). \tag{70}$$

By substitution of (68) into (70) we get components of the friction force vector

$$\begin{aligned} t^1 = -N[&C^{11}\cos\alpha_v + C^{12}\sin\alpha_v + C^{1111}\cos^3\alpha_v + (C^{1112}+C^{1121}\\ &+C^{1211})\cos^2\alpha_v\sin\alpha_v + (C^{1122}+C^{1212}+C^{1221})\sin^2\alpha_v\cos\alpha_v\\ &+C^{1222}\sin^3\alpha_v] \end{aligned} \tag{71}$$

$$\begin{aligned} t^2 = -N[&C^{21}\cos\alpha_v + C^{22}\sin\alpha_v + C^{2111}\cos^3\alpha_v + (C^{2112}+C^{2121}\\ &+C^{2211})\cos^2\alpha_v\sin\alpha_v + (C^{2122}+C^{2212}+C^{2221})\sin^2\alpha_v\cos\alpha_v\\ &+C^{2222}\sin^3\alpha_v]. \end{aligned} \tag{72}$$

The components C^{ij} and C^{ijkl} are taken with respect to the tensor basis formed by the orthogonal unit vectors.

The nonlinear model of frictional anisotropy has the following properties.

Property 1. The friction force equation of the form (59) satisfies the axiom of material objectivity.

According to the axiom of material objectivity the friction function (59) of the unit vector $\mathbf{v}$ and the scalar N must be form-invariant with respect to arbitrary transformation from the full orthogonal group $\mathcal{O}$, i.e.

$$\mathbf{t}(\mathbf{R}\mathbf{v}, N) = \mathbf{R}\mathbf{t}(\mathbf{v}, N), \quad \forall\, \mathbf{R}\in\mathcal{O}. \tag{73}$$

After substitution of (59) into (73) we get

$$\mathbf{t}(\mathbf{R}\mathbf{v}, N) = -N\mathbf{R}\{\mathbf{C}_1\mathbf{v} + [(\mathbf{C}_2\mathbf{v}\mathbf{R}^T)\mathbf{R}\mathbf{v}]\mathbf{v} + \cdots\} \equiv \mathbf{R}\mathbf{t}(\mathbf{v}, N). \tag{74}$$

The slip velocity and its unit vector are objective vectors. Thus, the friction force equation (59) satisfies the objectivity axiom. The relations (74) show that only terms with odd number of elements may be included in the polynomial (59).

Property 2. Any friction tensor is positive definite.

From the Second Law of Thermodynamics it follows that in every case of frictional contact the power of the friction force is nonpositive (Zmitrowicz, 1987)

$$\mathbf{t}\cdot\mathbf{V} \leqslant 0 \tag{75}$$

where $\mathbf{V}$ is the sliding velocity. Substituting eqn (59) into (75) and taking into account that N and $|\mathbf{V}|$ are positive and that

$$\mathbf{v} = \frac{\mathbf{V}}{|\mathbf{V}|} \tag{76}$$

we obtain the following condition

$$\mathbf{V}^T\mathbf{C}_1\mathbf{V}+\mathbf{V}^T[(\mathbf{C}_2\mathbf{v})\mathbf{v}]\mathbf{V} \geqslant 0 \tag{77}$$

for every $\mathbf{V}$. The inequality (77) can be replaced by two restrictions on the friction tensors $\mathbf{C}_1$ and $\mathbf{C}_2$

$$\mathbf{V}^T\mathbf{C}_1\mathbf{V} \geqslant 0 \tag{78}$$

$$\mathbf{V}^T(\mathbf{V}^T\mathbf{C}_2\mathbf{V})\mathbf{V} \geqslant 0. \tag{79}$$

After decomposition of the tensor $\mathbf{C}_1$ into a symmetric and an antisymmetric part it is seen that (78) is equal to zero for the antisymmetric part of the tensor. The inequality (78) holds for the symmetric part of the tensor $\mathbf{C}_1$ if all main diagonal minors of the matrix representation of the symmetric tensor are positive,

$$C^{11} \geqslant 0 \tag{80}$$

$$\det\left[\tfrac{1}{2}(\mathbf{C}_1+\mathbf{C}_1^T)\right] \geqslant 0. \tag{81}$$

It can be replaced by the condition that every eigenvalue of the symmetric part of the tensor $\mathbf{C}_1$ is non-negative and it has at least one nonzero eigenvalue.

Property 3. Friction with four, three, two, one or zero principal directions is determined by the nonlinear friction equation (59) with two terms of the polynomial.

When sliding in a principal direction, the friction force vector is related to this direction ($\mathbf{v}$) by

$$\mathbf{t} = -\mu N\mathbf{v} \tag{82}$$

where the vector $\mathbf{t}$ is defined by (59). In component notation referring to the orthogonal system it has the form

$$t^1 = -\mu N\cos\alpha_v \tag{83}$$

$$t^2 = -\mu N\sin\alpha_v \tag{84}$$

where μ is a friction coefficient for sliding in a principal direction, α_v is an angle determining the principal direction with respect to the reference system. After multiplication of eqn (83) by $\sin\alpha_v$, of eqn (84) by $\cos\alpha_v$ and subtraction of these equations, we obtain

$$t^1\sin\alpha_v - t^2\cos\alpha_v = 0. \tag{85}$$

Substituting the friction force components (71) and (72) into this relation yields

$$
\begin{aligned}
& C^{2111}\cos^4\alpha_r - C^{1222}\sin^4\alpha_r + (C^{2112}+C^{2121}+C^{2211}-C^{1111}) \\
& \quad \times \cos^3\alpha_r \sin\alpha_r + (C^{2222}-C^{1122}-C^{1212}-C^{1221}) \\
& \quad \times \sin^3\alpha_r \cos\alpha_r + (C^{2122}+C^{2212}+C^{2221}-C^{1112}-C^{1121}-C^{1211}) \\
& \quad \times \cos^2\alpha_r \sin^2\alpha_r + C^{21}\cos^2\alpha_r - C^{12}\sin^2\alpha_r \\
& \quad + (C^{22}-C^{11})\sin\alpha_r\cos\alpha_r = 0.
\end{aligned}
\tag{86}
$$

It is a fourth-order trigonometrical equation for the principal directions of friction. Depending on the C^{ijkl} and C^{ij} values it can have four, three, two, one or zero real roots for $\alpha_r \in \langle 0, \pi \rangle$. Thus, the frictional anisotropies, with four, three, two, one and zero principal directions of friction can be distinguished. Generally, the polynomial (59) defines frictional anisotropies with arbitrary number of principal directions. The linear model of frictional anisotropy is limited to a maximum of two principal directions.

Relying on the experimental investigations of frictional anisotropy of single crystals (Seal, 1957; Bailey and Gwathmey, 1962; Bowden and Brooks, 1966; Buckley, 1968; Casey and Wilks, 1973; Enomato and Tabor, 1980), one would expect that, in some cases, the number of principal directions of friction can be finite and larger than two. This property is completely described by the nonlinear model of frictional anisotropy.

Property 4. Anisotropic, tetragonal anisotropic, orthotropic and isotropic friction can be distinguished with the aid of the friction tensor $\mathbf{C}_2$ and symmetry groups.

The tensors $\mathbf{C}_1$ and $\mathbf{C}_2$ define the frictional anisotropy and symmetries. Due to the polynomial character of the constitutive relation (59), the symmetry groups of the tensors $\mathbf{C}_1$ and $\mathbf{C}_2$ are complementary. In other words the symmetry group of the friction equation (59) is the intersection of the symmetry groups of the tensors $\mathbf{C}_1$ and $\mathbf{C}_2$

$$\mathscr{G}_t = \mathscr{G}_{\mathbf{C}_1, \mathbf{C}_2} = \mathscr{G}_{\mathbf{C}_1} \cap \mathscr{G}_{\mathbf{C}_2}. \tag{87}$$

A subgroup $\mathscr{G}_{\mathbf{C}_2}$ of the full orthogonal group $\mathscr{O}$ is called a symmetry group of the tensor $\mathbf{C}_2$ if it satisfies the following relation

$$\mathscr{G}_{\mathbf{C}_2} = \left\{ \mathbf{R} : \mathbf{R} \in \mathscr{O}, \left(\overset{4}{\underset{1}{\otimes}} \mathbf{R} \right) \cdot \mathbf{C}_2 = \mathbf{C}_2 \right\} \tag{88}$$

where, $\overset{4}{\underset{1}{\otimes}}$ denotes the composition

$$\left(\overset{4}{\underset{1}{\otimes}} \mathbf{R} \right) \cdot \mathbf{C}_2 = C^{ijkl}\mathbf{R}\mathbf{k}_i \otimes \mathbf{R}\mathbf{e}_j \otimes \mathbf{R}\mathbf{e}_k \otimes \mathbf{R}\mathbf{e}_l. \tag{89}$$

The definition (88) says that every member of the symmetry group commutes with $\mathbf{C}_2$. For convenience and without loss of generality we assume that the tensor $\mathbf{C}_1$ is a spherical tensor. Then, the symmetry group of the tensor $\mathbf{C}_2$ determines symmetry of the constitutive relation (59)

$$\mathscr{G}_{\mathbf{C}_1, \mathbf{C}_2} = \mathscr{G}_{\mathbf{C}_2} \tag{90}$$

since

$$\mathscr{G}_{\mathbf{C}_1} = \mathscr{O}. \tag{91}$$

A classification of the nonlinear models of frictional anisotropy can be carried out with the help of symmetry groups.

The group $\mathscr{G}_{C_2}$ of symmetry of tetragonal anisotropic friction has a four-fold rotation axis normal to the contact and the subgroup of mirror reflections with respect to subspaces orthogonal to four principal directions of friction. The generators of this eight-order symmetry group are

$$\mathscr{G}_{C_2} = \{\mathbf{1}, \mathbf{R}_{\mathbf{n}}^{\pi/2}, \mathbf{J}_{\mathbf{m}_1}, \mathbf{J}_{\mathbf{m}_2}, \mathbf{J}_{\mathbf{m}_3}, \mathbf{J}_{\mathbf{m}_4}\}. \tag{92}$$

Here, the symbol $\mathbf{R}_{\mathbf{n}}^{\varphi}$ denotes a rotation about the n-fold rotation axis with the unit vector $\mathbf{n}$ through an angle being a multiplicity of

$$\varphi = \frac{2\pi}{n}, \quad n = 1, 2, 3, \ldots \tag{93}$$

For every rotation tensor $\mathbf{R}_{\mathbf{n}}^{\varphi}$ there exists an orthonormal basis $\{\mathbf{n}, \mathbf{k}_i\}$, $i = 1, 2$ and a number $\varphi \in \langle 0, 2\pi \rangle$ such that

$$\mathbf{R}_{\mathbf{n}}^{\varphi} = \mathbf{n} \otimes \mathbf{n} + \cos\varphi(\mathbf{k}_1 \otimes \mathbf{k}_1 + \mathbf{k}_2 \otimes \mathbf{k}_2) + \sin\varphi\,(\mathbf{k}_1 \otimes \mathbf{k}_2 - \mathbf{k}_2 \otimes \mathbf{k}_1) \tag{94}$$

where

$$\mathbf{k}_i \cdot \mathbf{k}_j = \delta_{ij}, \quad \mathbf{n} \cdot \mathbf{k}_i = 0, \quad \mathbf{n} \cdot \mathbf{n} = 1, \quad i, j = 1, 2. \tag{95}$$

The central inversion transformation is contained in the subgroup $\{\mathbf{R}_{\mathbf{n}}^{\pi/2}\}$. Table 1 shows composition operations for the generators of the symmetry group of the tetragonal anisotropy (92). The rotation angles $\pi/2$, $3\pi/2$ and inversion $-\mathbf{1}$ define the elements of the subgroup of rotation about four-fold rotation axes. From Table 1 it is seen that the group of symmetry (92) is a non-Abelian group.

The symmetry conditions allow us to determine the restrictions imposed on $\mathbf{C}_2$ for various symmetries corresponding to different kinds of anisotropy. It means that symmetries are equivalent to specific relations between tensor coefficients. To describe the transformations which characterize a class of symmetry, it is convenient to choose a reference coordinate system. Let $0xy$ be the orthogonal frame system and the base vectors $\{\mathbf{k}_i\}$ and $\{\mathbf{e}_i\}$ coincide with this system. Sixteen elements of the representation of the tensor $\mathbf{C}_2$ can

Table 1. Cayley square for the generators of the symmetry group of the tetragonal anisotropic friction.

	$\mathbf{1}$	$\frac{\pi}{2}$	$-\mathbf{1}$	$\frac{3\pi}{2}$	$\mathbf{J}_{m_1}$	$\mathbf{J}_{m_2}$	$\mathbf{J}_{m_3}$	$\mathbf{J}_{m_4}$
$\mathbf{1}$	$\mathbf{1}$	$\frac{\pi}{2}$	$-\mathbf{1}$	$\frac{3\pi}{2}$	$\mathbf{J}_{m_1}$	$\mathbf{J}_{m_2}$	$\mathbf{J}_{m_3}$	$\mathbf{J}_{m_4}$
$\frac{\pi}{2}$	$\frac{\pi}{2}$	$-\mathbf{1}$	$\frac{3\pi}{2}$	$\mathbf{1}$	$\mathbf{J}_{m_4}$	$\mathbf{J}_{m_1}$	$\mathbf{J}_{m_2}$	$\mathbf{J}_{m_3}$
$-\mathbf{1}$	$-\mathbf{1}$	$\frac{3\pi}{2}$	$\mathbf{1}$	$\frac{\pi}{2}$	$\mathbf{J}_{m_3}$	$\mathbf{J}_{m_4}$	$\mathbf{J}_{m_1}$	$\mathbf{J}_{m_2}$
$\frac{3\pi}{2}$	$\frac{3\pi}{2}$	$\mathbf{1}$	$\frac{\pi}{2}$	$-\mathbf{1}$	$\mathbf{J}_{m_2}$	$\mathbf{J}_{m_3}$	$\mathbf{J}_{m_4}$	$\mathbf{J}_{m_1}$
$\mathbf{J}_{m_1}$	$\mathbf{J}_{m_1}$	$\mathbf{J}_{m_2}$	$\mathbf{J}_{m_3}$	$\mathbf{J}_{m_4}$	$\mathbf{1}$	$\frac{\pi}{2}$	$-\mathbf{1}$	$\frac{3\pi}{2}$
$\mathbf{J}_{m_2}$	$\mathbf{J}_{m_2}$	$\mathbf{J}_{m_3}$	$\mathbf{J}_{m_4}$	$\mathbf{J}_{m_1}$	$\frac{3\pi}{2}$	$\mathbf{1}$	$\frac{\pi}{2}$	$-\mathbf{1}$
$\mathbf{J}_{m_3}$	$\mathbf{J}_{m_3}$	$\mathbf{J}_{m_4}$	$\mathbf{J}_{m_1}$	$\mathbf{J}_{m_2}$	$-\mathbf{1}$	$\frac{3\pi}{2}$	$\mathbf{1}$	$\frac{\pi}{2}$
$\mathbf{J}_{m_4}$	$\mathbf{J}_{m_4}$	$\mathbf{J}_{m_1}$	$\mathbf{J}_{m_2}$	$\mathbf{J}_{m_3}$	$\frac{\pi}{2}$	$-\mathbf{1}$	$\frac{3\pi}{2}$	$\mathbf{1}$

be arranged in a table

$$[\mathbf{C}_2] = \begin{array}{c} \begin{array}{cccc} 11 & 22 & 21 & 12 \end{array} \\ \left[\begin{array}{cccc} C^{1111} & C^{1122} & C^{1121} & C^{1112} \\ C^{2211} & C^{2222} & C^{2221} & C^{2212} \\ C^{2111} & C^{2122} & C^{2121} & C^{2112} \\ C^{1211} & C^{1222} & C^{1221} & C^{1212} \end{array}\right] \end{array} \begin{array}{c} \\ 11 \\ 22 \\ 21 \\ 12 \end{array} \tag{96}$$

They are ordered by pairs of indices. The restriction on the components of the tensor $\mathbf{C}_2$ due to the symmetry conditions can be obtained by a standard procedure, e.g. an inspection method (Nye, 1957). In this way the following four independent elements of the tensor $\mathbf{C}_2$ for the tetragonal anisotropy are determined.

$$[\mathbf{C}_2] = \begin{array}{c} \begin{array}{cccc} 11 & 22 & 21 & 12 \end{array} \\ \left[\begin{array}{cccc} C^{1111} & C^{1122} & 0 & 0 \\ C^{1122} & C^{1111} & 0 & 0 \\ 0 & 0 & C^{2121} & C^{2112} \\ 0 & 0 & C^{2112} & C^{2121} \end{array}\right] \end{array} \begin{array}{c} \\ 11 \\ 22 \\ 21 \\ 12 \end{array} \tag{97}$$

Substituting this representation into (71), (72) and (86) we get the friction force components and principal directions for the tetragonal anisotropic friction.

Orthotropic friction has been defined by the symmetry group which contains the trivial subgroup and the subgroup of two mirror reflections (40). It can be shown that in the case of orthotropy the representation of the tensor $\mathbf{C}_2$ has two independent elements specified by

$$[\mathbf{C}_2] = \begin{array}{c} \begin{array}{cccc} 11 & 22 & 21 & 12 \end{array} \\ \left[\begin{array}{cccc} C^{1111} & 0 & 0 & 0 \\ 0 & C^{2222} & 0 & 0 \\ 0 & 0 & C^{2222} & 0 \\ 0 & 0 & 0 & C^{1111} \end{array}\right] \end{array} \begin{array}{c} \\ 11 \\ 22 \\ 21 \\ 12 \end{array} \tag{98}$$

Using this tensor representation we get two mutually orthogonal principal directions of friction and an ellipse as the friction force hodograph.

The isotropic friction is defined by the full orthogonal group (47). In this case the representation of the tensor $\mathbf{C}_2$ has one independent element and the table (96) reduces to

$$[\mathbf{C}_2] = \begin{array}{c} \begin{array}{cccc} 11 & 22 & 21 & 12 \end{array} \\ \left[\begin{array}{cccc} C^{1111} & 0 & 0 & 0 \\ 0 & C^{1111} & 0 & 0 \\ 0 & 0 & C^{1111} & 0 \\ 0 & 0 & 0 & C^{1111} \end{array}\right] \end{array} \begin{array}{c} \\ 11 \\ 22 \\ 21 \\ 12 \end{array} \tag{99}$$

Here, all sliding directions are principal directions of friction and the friction force hodograph is a circle.

There is a particular case of frictional anisotropy. It corresponds to a constant angle of inclination of the friction force vector to the sliding direction for any direction

$$\beta(\alpha_x) = \text{const.} \tag{100}$$

Rotations and trivial subgroup built its symmetry group. The hodograph is a circle as in the isotropic case. Then the representation of the tensor $\mathbf{C}_2$ is given by

$$[\mathbf{C}_2] = \begin{array}{c} \begin{array}{cccc} 11 & 22 & 21 & 12 \end{array} \\ \left[\begin{array}{cccc} C^{1111} & 0 & C^{1121} & 0 \\ 0 & C^{1111} & 0 & -C^{1121} \\ -C^{1121} & 0 & C^{1111} & 0 \\ 0 & C^{1121} & 0 & C^{1111} \end{array}\right] \end{array} \begin{array}{c} \\ 11 \\ 22 \\ 21 \\ 12 \end{array} \tag{101}$$

The same situation occurs in the linear model of frictional anisotropy. Then the second-order friction tensor $\mathbf{C}_1$ of the form (8) has the following components

$$C^{11} = C^{22} \tag{102}$$

$$C^{12} = -C^{21}. \tag{103}$$

The trivial subgroup (45) is a symmetry group in other cases of the nonlinear model of frictional anisotropy and there are no restrictions on the representation of the friction tensor $\mathbf{C}_2$.

With the aid of the next terms of the polynomial (59) we can describe, e.g. a hexagonal and octogonal anisotropy of friction. They are determined by symmetry groups with six-fold and eight-fold rotation axes and mirror reflection subgroups with respect to six and eight principal directions, respectively. The method to be followed in the tensor representation determination is analogous to that which has been employed in the tetragonal anisotropic case. It is possible that there is a close relation between the friction phenomenon on a crystal face and a crystallographic system of a given compound. A dependence of the friction symmetry and a dependence of the number of principal directions of friction on the crystal symmetry are not known, yet.

Property 5. All linear and nonlinear models of frictional anisotropy described by the friction tensor $\mathbf{C}_1$ and $\mathbf{C}_2$ are centrosymmetrical.

The inversion is that transformation which characterizes anisotropy with centre of symmetry. For the two-dimensional space the central inversion transformation is equivalent to the rotation about the normal $\mathbf{n}$ to the space of the angle π

$$-\mathbf{1} \equiv \mathbf{R}_{\mathbf{n}}^{\pi}. \tag{104}$$

Taking the definition of symmetry group of the tensor $\mathbf{C}_1$ (36) and $\mathbf{C}_2$ (88) we see that the following relations hold.

$$\left[\overset{2}{\underset{1}{\otimes}}(-\mathbf{1})\right]\cdot\mathbf{C}_1 = C^{ij}(-\mathbf{1})\mathbf{k}_i \otimes (-\mathbf{1})\mathbf{e}_j = \mathbf{C}_1 \tag{105}$$

$$\left[\overset{4}{\underset{1}{\otimes}}(-\mathbf{1})\right]\cdot\mathbf{C}_2 = C^{ijkl}(-\mathbf{1})\mathbf{k}_i \otimes (-\mathbf{1})\mathbf{e}_j \otimes (-\mathbf{1})\mathbf{e}_k \otimes (-\mathbf{1})\mathbf{e}_l = \mathbf{C}_2. \tag{106}$$

Thus, the inversion transformation does not change the friction tensors $\mathbf{C}_1$ and $\mathbf{C}_2$, and both groups of symmetry $\mathscr{G}_{\mathbf{C}_1}$ and $\mathscr{G}_{\mathbf{C}_2}$ are centrosymmetrical. They contain the central inversion transformation and its products with other group transformations. In view of these results the linear and nonlinear models of frictional anisotropy are said to be centrosymmetrical. The existence of a centre of symmetry has no effect on the number of tensor components.

In accordance with this property, if we change the sliding direction by an angle π then the friction force changes its direction by the same angle but does not change its magnitude and its angle of inclination β. Thus, the absolute value of the anisotropic friction force depends on the sliding direction and does not depend on the sense of this direction.

The following two general properties hold in every case of the linear and nonlinear model of frictional anisotropy. Frictional description constructed in the space $\mathscr{R}^2$ is form-invariant with respect to the mirror reflection related to that space. Frictional anisotropy homogeneous over the contact area has an Abelian continuous symmetry group of translations in the space $\mathscr{R}^2$.

Tangential forces to the contact pointing in the area bounded by the nonlinear friction force hodograph define the adherence region. A similar condition has been formulated in the case of linear frictional anisotropy. The initial direction of motion is such that the sum of components orthogonal to it of the tangential force causing it (f) and the friction force is equal to zero:

$$[\mathbf{f}-(\mathbf{f}\cdot\mathbf{v})\mathbf{v}]+[\mathbf{t}-(\mathbf{t}\cdot\mathbf{v})\mathbf{v}] = \mathbf{0}. \tag{107}$$

Then the absolute value of the tangential force projection on the motion direction is greater than the absolute value of the friction force projection on this direction:

$$|\mathbf{f}\cdot\mathbf{v}| > |\mathbf{t}\cdot\mathbf{v}|. \tag{108}$$

We shall illustrate some properties of the nonlinear model of frictional anisotropy by diagrams in Appendix B.

4. A COMPOSITION OF TWO DIFFERENT FRICTIONAL ANISOTROPIES

Up to now our analysis of the linear and nonlinear models of frictional anisotropy has been related to friction forces during the sliding of two contacting surfaces with single isotropic and anisotropic frictional properties. Zmitrowicz (1978 and 1981) considered frictional forces during the sliding of surfaces with different anisotropic friction properties. Now, we expand it to the nonlinear model.

Assume that there are orthogonal reference systems on both contacting surfaces. The coefficients of the tensors $\mathbf{C}_1$ and $\mathbf{C}_2$ can be determined experimentally by sliding a third body with isotropic friction properties on the surfaces of each contacting body. The values $\overset{(s)}{C}{}^{ij}$, $\overset{(s)}{C}{}^{ijkl}$ $(s = 1, 2)$ found experimentally are used to formulate matrix representations of the friction tensors $\overset{(s)}{\mathbf{C}}_1$, $\overset{(s)}{\mathbf{C}}_2$. These tensors describe the friction properties of surfaces (1) and (2), respectively.

Relative positions of the contacting surfaces are described by an angle φ (Fig. 3). Then

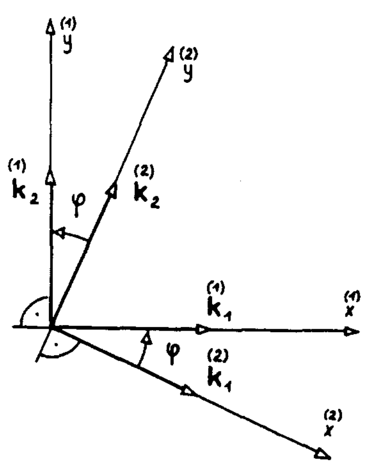

Fig. 3. Relative position of the reference systems at the contact of surfaces (1) and (2).

the following relation holds between the unit vectors of the bases of the reference systems

$$\overset{(2)}{\mathbf{k}}_i = B_i^j \overset{(1)}{\mathbf{k}}_j \quad i, j = 1, 2 \tag{109}$$

where

$$[B_i^j] = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} \equiv [\mathbf{B}] \tag{110}$$

is an orthogonal rotation matrix.

Hypothesis 2. Let us assume that for a given normal pressure the friction force on the contact surface during relative sliding is equal to the product of a "composition coefficient" and the sum of the friction forces obtained for each surface taken separately, i.e.

$$\mathbf{t} = \mathscr{K}\left(\overset{(1)}{\mathbf{t}} + \overset{(2)}{\mathbf{t}}\right) \tag{111}$$

The forces $\overset{(1)}{\mathbf{t}}$ and $\overset{(2)}{\mathbf{t}}$ correspond to friction when sliding a third body with isotropic friction properties along the contacting surfaces. The coefficient $\mathscr{K}$, which is called the composition coefficient, is an experimental quantity and its value will not affect the description of the directional properties of friction in the contact of surfaces.

After substituting the friction constitutive equation (59) and the transformation relation (109) into eqn (111), we obtain the friction force relative to the contact of two different surfaces.

$$\mathbf{t} = -N(\tilde{\mathbf{C}}_1\mathbf{v} + \tilde{\mathbf{C}}_2\mathbf{v}^3 + \cdots). \tag{112}$$

According to the definition (111), the matrix representations of the friction tensors $\tilde{\mathbf{C}}_1$ and $\tilde{\mathbf{C}}_2$ are defined by

$$[\tilde{\mathbf{C}}_1] = \mathscr{K}\left[\overset{(1)}{\mathbf{C}}_1 + \mathbf{B}^T\overset{(2)}{\mathbf{C}}_1\mathbf{B}\right] \tag{113}$$

$$[\tilde{\mathbf{C}}_2] = \mathscr{K}\left[\overset{(1)}{\mathbf{C}}_2 + \mathbf{B}^T\left(\mathbf{B}^T\overset{(2)}{\mathbf{C}}_2\mathbf{B}\right)\mathbf{B}\right] \tag{114}$$

where $\left[\overset{(1)}{\mathbf{C}}_1\right]$, $\left[\overset{(2)}{\mathbf{C}}_1\right]$, $\left[\overset{(1)}{\mathbf{C}}_2\right]$ and $\left[\overset{(2)}{\mathbf{C}}_2\right]$ are matrix representations of the friction tensors for the surfaces (1) and (2), respectively.

Experimental measurements of the composition anisotropy for two rough surfaces with orthotropic friction properties are given by Sharpin (1957). A change of frictional anisotropy with respect to a relative orientation of faces of contacting diamond crystals has been observed by Seal (1957).

5. CONCLUSIONS

(1) A linear model of frictional anisotropy with two principal directions of friction and with elliptical hodographs of the friction force is presented in this paper. Frictional isotropy has an infinite number of principal directions and a circle for hodograph.

(2) A nonlinear model of frictional anisotropy describing the friction with an arbitrary number of principal directions of friction and with arbitrary shapes of the friction force hodographs has been formulated.

(3) The experiments show that there are classes of friction problems where the linear and nonlinear models of frictional anisotropy are relevant. It might be expected that future

investigations will give new experimental data on anisotropic friction. Then, adequate quantitative comparison of experimental and theoretical results will be possible.

(4) The frictional anisotropy models can be extended to include other contact phenomena. Friction tensor representations can be functions of surface temperature, temperature gradients, sliding velocity, normal pressure, etc. (Zmitrowicz, 1987).

Considering frictional anisotropy in contact problems, vibration and dynamical analysis, tribological experiments, machining processes, metal forming, rail/wheel contact mechanics, etc., leads to a more realistic image of these physical processes Frictional anisotropy can play a great role in engineering applications of crystals. For instance diamonds are used for boring tools, abrasive disks and glass cutting. Rubies are used for bearings in precision measuring instruments. Finally, diamond, sapphire and SiC are used as so-called wear-resistant materials.

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APPENDIX A. ILLUSTRATION OF THE LINEAR MODEL OF ANISOTROPIC FRICTION

In accordance with (19) we may take any real numbers as the friction coefficients μ_{ij} with only one restriction, namely the friction tensor $\mathbf{C}_1$ must be positive definite. As an example we take coefficient values which are typical for friction in metals. We consider the following cases: anisotropic friction without principal directions ($\mu_{11} = 0.06$; $\mu_{12} = -0.03$; $\mu_{21} = 0.01$; $\mu_{22} = 0.07$; $\varepsilon_x = \varepsilon_y = 0^\circ$), anisotropic friction with one principal direction ($\mu_{11} = 0.1$; $\mu_{12} = -0.02$; $\mu_{21} = 0.02$; $\mu_{22} = 0.06$; $\varepsilon_x = \varepsilon_y = 0^\circ$), anisotropic friction with two principal directions ($\mu_{11} = 0.1$; $\mu_{12} = 0.04$; $\mu_{21} = 0.01$; $\mu_{22} = 0.1$; $\varepsilon_x = \varepsilon_y = 0^\circ$), orthotropic friction ($\mu_{11} = 0.13$; $\mu_{12} = -0.03$; $\mu_{21} = -0.03$; $\mu_{22} = 0.08$; $\varepsilon_x = \varepsilon_y = 0^\circ$). Figures A1–A4 show the following plots: (a) the friction coefficient $\mu_\tau^\perp$ and (b) friction coefficient μ_τ referring to polar coordinates, (c) the inclination angle β and (d) the friction force hodograph with respect to an orthogonal reference system. Principal directions of friction and corresponding zero points of the function $\beta = f(\alpha_\tau)$ are shown in the figures. The presented diagrams cover all types of frictional anisotropy.

In the case of frictional orthotropy (Fig. A4) the directions of $\mu_\tau^\parallel$ extreme values (16) coincide with the principal directions of frictions and with the hodograph ellipse axes. The direction of maximal value of $\mu_\tau^\parallel$ coincides with the longer axis of the hodograph ellipse. It is a typical property of frictional orthotropy.

Experimental measurements presented by Rabinowicz (1957), Sharpin (1957) and Halaunbrenner (1960) provide a good illustration of the linear model of frictional anisotropy, e.g. frictional orthotropy is dealt with by Rabinowicz (1957). Experiments by Halaunbrenner (1960) are typical of frictional anisotropy without principal directions.

It is important to consider the directional dependence of friction between contacting bodies in relative motion, because it changes markedly the nature of the phenomenon. We analyse properties of a material point motion in a plane with frictional anisotropy. Figures A5–A7 present trajectories of the material point motion in the plane with frictional properties shown by the Figs A1, A2 and A4, respectively. The following equation describes the motion

$$m\ddot{\mathbf{q}} = \mathbf{t} \tag{A1}$$

where, m is the mass of the material point, $\mathbf{q}$ its position vector and $\mathbf{t}$ the friction force vector. The velocity $\dot{\mathbf{q}}_0$ is taken as the initial condition of motion. We consider different directions of the initial velocity. They are given by the angle α_0

$$[\dot{\mathbf{q}}_0] = [\dot{q}_0 \cos \alpha_0, \dot{q}_0 \sin \alpha_0]^T \tag{A2}$$

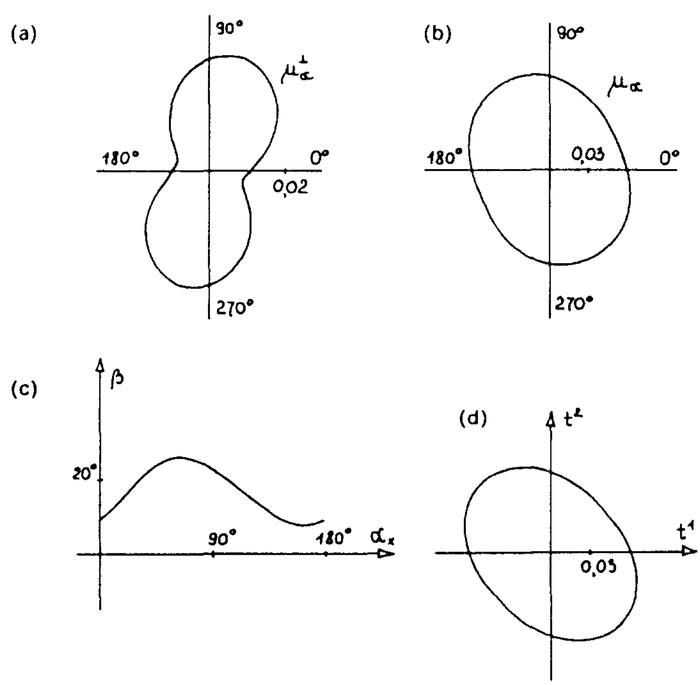


Fig. A1. Illustration of the anisotropic friction without principal directions of friction: (a) friction coefficient $\mu_\tau^\perp$; (b) friction coefficient μ_τ; (c) inclination angle β; (d) friction force hodograph.

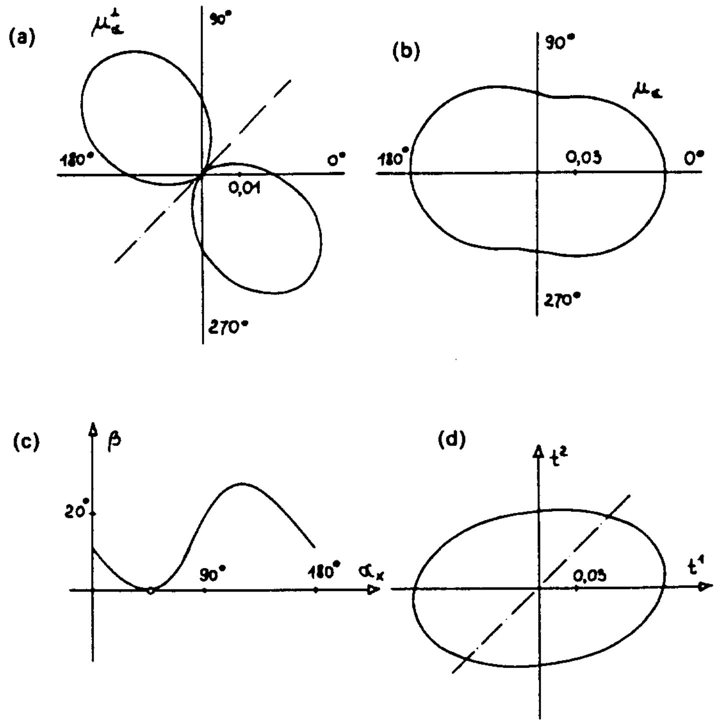


Fig. A2. Illustration of the anisotropic friction with one principal direction of friction: (a) friction coefficient $\mu_\alpha^\perp$; (b) friction coefficient μ_α; (c) inclination angle β; (d) friction force hodograph.

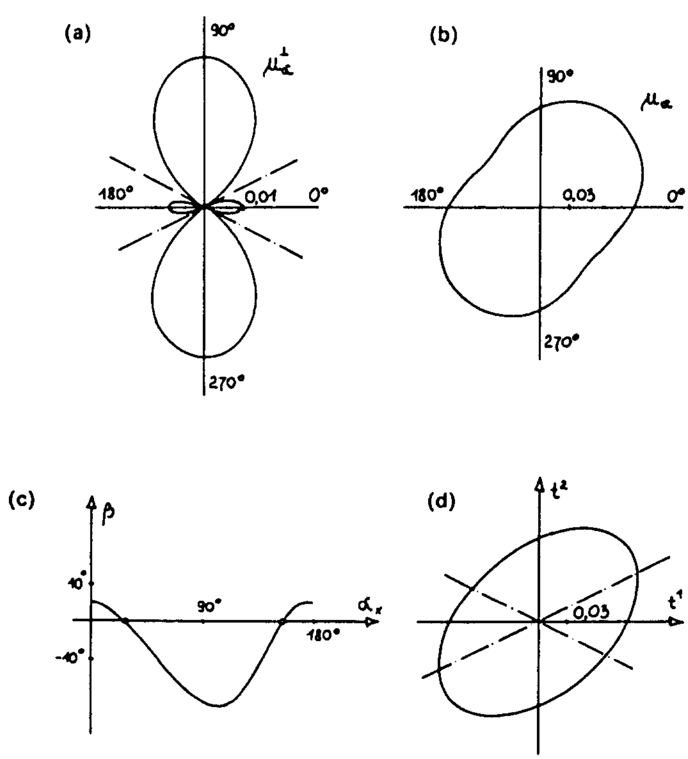


Fig. A3. Illustration of the anisotropic friction with two principal directions of friction: (a) friction coefficient $\mu_\alpha^\perp$; (b) friction coefficient μ_α; (c) inclination angle β; (d) friction force hodograph.

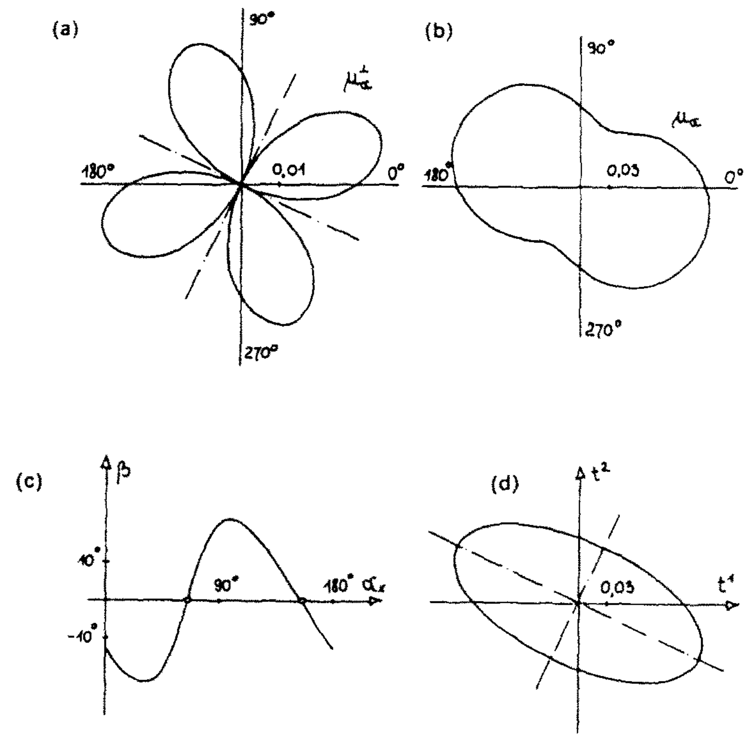
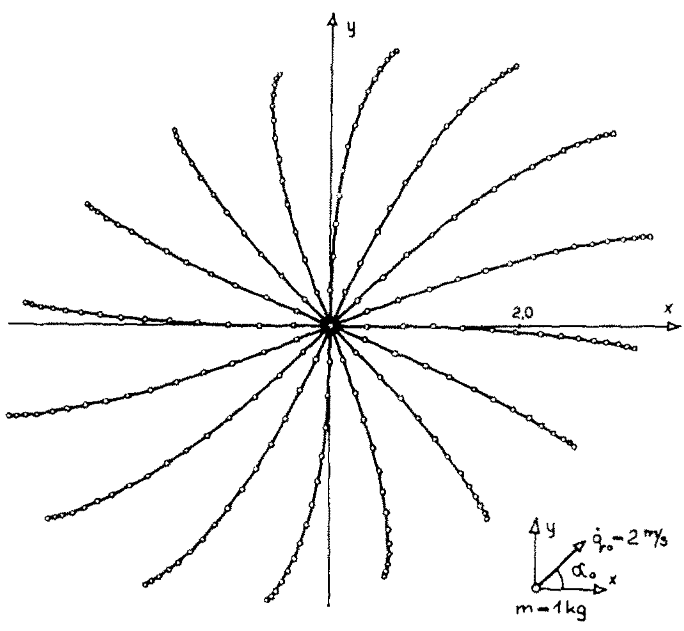
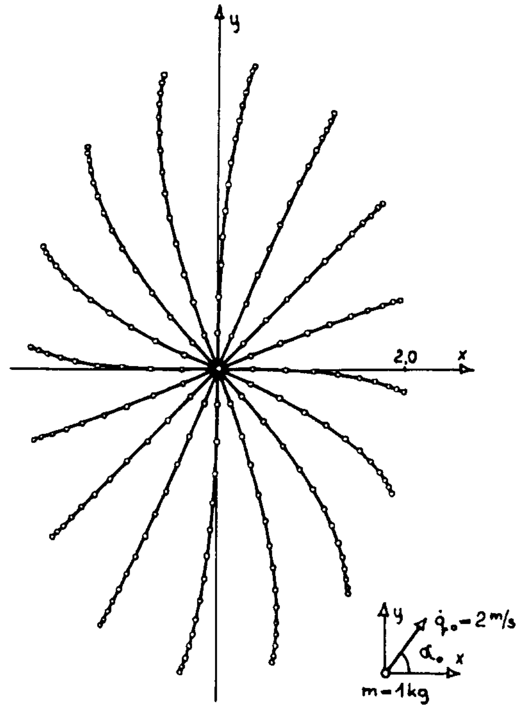


Fig. A4. Illustration of the orthotropic friction: (a) friction coefficient $\mu_\alpha^\perp$; (b) friction coefficient μ_α; (c) inclination angle β; (d) friction force hodograph.



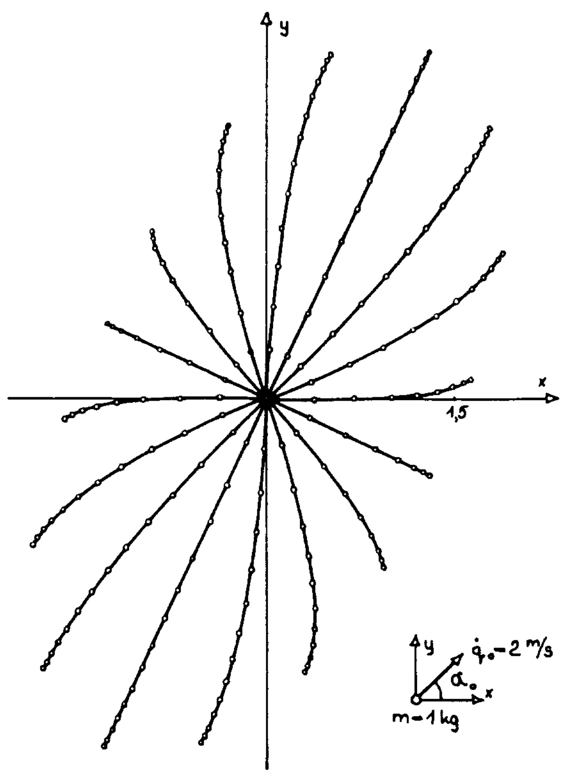
α_0	0°	22,5°	45°	67,5°	90°	112,5°	135°	157,5°
	180°	202,5°	225°	247,5°	270°	292,5°	315°	337,5°

Fig. A5. Motion of a material point in a plane with anisotropic friction without principal directions of friction.



α_0	0°	22,5°	45°	67,5°	90°	112,5°	135°	157,5°
	180°	202,5°	225°	247,5°	270°	292,5°	315°	337,5°

Fig. A6. Motion of a material point in a plane with anisotropic friction with one principal direction of friction.



α_0	20°	42,5°	65°	87,5°	110°	132,5°	155°	177,5°
	200°	222,5°	245°	267,5°	290°	312,5°	335°	357,5°

Fig. A7. Motion of a material point in a plane with orthotropic friction.

and specified in Figs A5–A7. Components of the slip velocity unit vector are as follows

$$v_1 = \frac{\dot{q}_1}{\sqrt{(\dot{q}_1)^2 + (\dot{q}_2)^2}}, \quad v_2 = \frac{\dot{q}_2}{\sqrt{(\dot{q}_1)^2 + (\dot{q}_2)^2}} \tag{A3}$$

where $\dot{q}_1$ and $\dot{q}_2$ are the velocity components. The nonlinear equation (A1) is solved by means of the Runge–Kutta fourth-order method.

The material point motion is a retarded motion. The length of the point trajectory depends on the frictional resistance. Intervals between points on the trajectory shown in Figs A5–A7 correspond to constant time intervals (0.2 s). The trajectory is a segment colinear with the principal direction of friction when the motion occurs in this direction. The trajectories in the case of orthotropic friction are curved in the direction of minimal friction.

APPENDIX B. ILLUSTRATION OF THE NONLINEAR MODEL OF ANISOTROPIC FRICTION

The following examples of the friction nonlinear model are considered: anisotropic friction with three principal directions ($C^{11} = C^{22} = 0.1$; $C^{1111} = C^{2121} = 0.05$; $C^{1122} = -C^{2222} = -C^{1212} = -0.02$; $C^{1121} = 0.03$; $C^{2112} = -0.06$; $C^{1222} = -0.01$), anisotropic friction with four principal directions ($C^{11} = C^{22} = 0.07$; $C^{1111} = C^{2121} = 0.05$; $C^{1122} = C^{2112} = 0.03$; $C^{2211} = 0.06$; $C^{2222} = C^{1221} = C^{1212} = 0.02$) and tetragonal anisotropic friction ($C^{11} = C^{22} = 0.07$; $C^{1111} = C^{2222} = 0.02$; $C^{1122} = C^{2211} = 0.08$; $C^{2121} = C^{1212} = 0.01$; $C^{2112} = C^{1221} = 0.06$). Only the nonzero elements of the tensors $\mathbf{C}_1$ and $\mathbf{C}_2$ are given. Figures B1–B3 show the diagrams of the friction coefficient $\mu_\tau^\perp$ (a), the friction coefficient μ_τ (b), the inclination angle β (c) and the friction force hodograph (d). It is seen that the nonlinear model of frictional anisotropy gives hodographs of complex shapes. Figures B4 and B5 present trajectories of the material point motion in the surface with frictional anisotropies described by the nonlinear models. The frictional properties of the surface are shown in Figs B2 and B3, respectively.

A similarity of theoretical and experimental results can be noticed if we compare the plots for the tetragonal anisotropy (Fig. B3) with experimental investigations of frictional anisotropy for diamond crystals (Bowden and Brooks, 1966; Casey and Wilks, 1973; Enomoto and Tabor, 1980) and for rough surfaces (Zieliński, 1964). Experiments carried out by Zieliński (1964) show that the friction force hodograph can be different from a circle and an ellipse. It results from the complex influence of the surface roughness produced by machining on the frictional anisotropy.

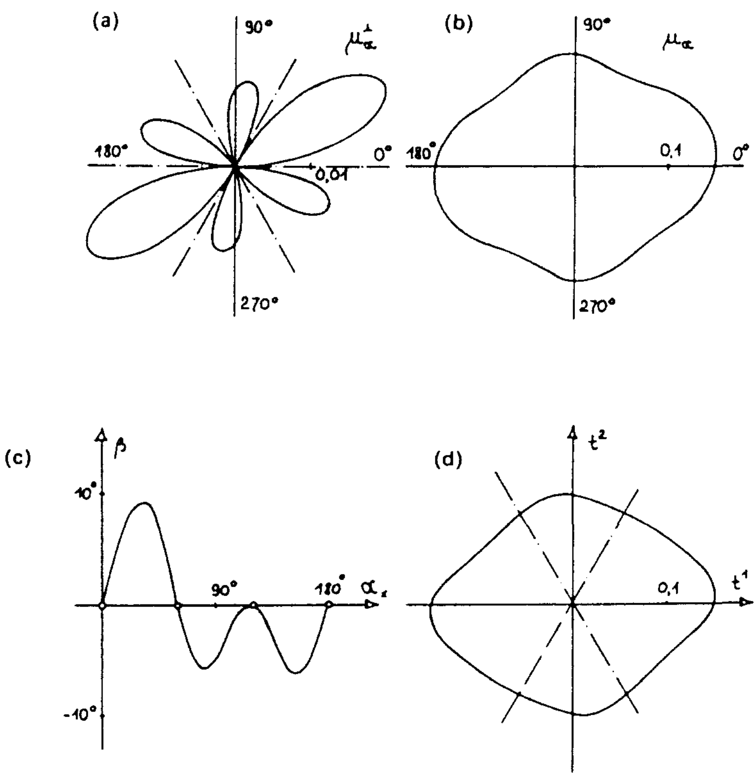


Fig. B1. Illustration of the anisotropic friction with three principal directions of friction: (a) friction coefficient $\mu_\tau^\perp$; (b) friction coefficient μ_τ; (c) inclination angle β; (d) friction force hodograph.

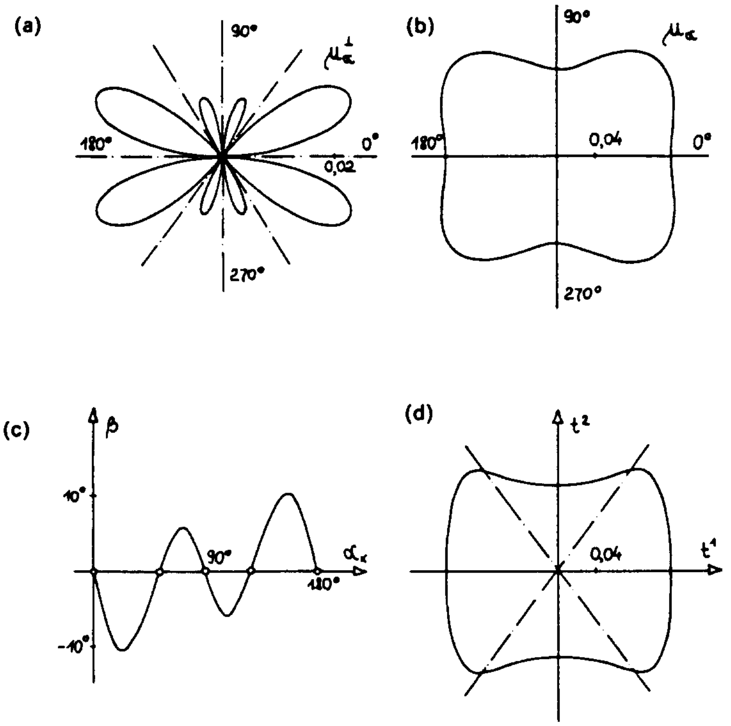


Fig. B2. Illustration of the anisotropic friction with four principal directions of friction: (a) friction coefficient $\mu_\alpha^\perp$; (b) friction coefficient μ_α; (c) inclination angle β; (d) friction force hodograph.

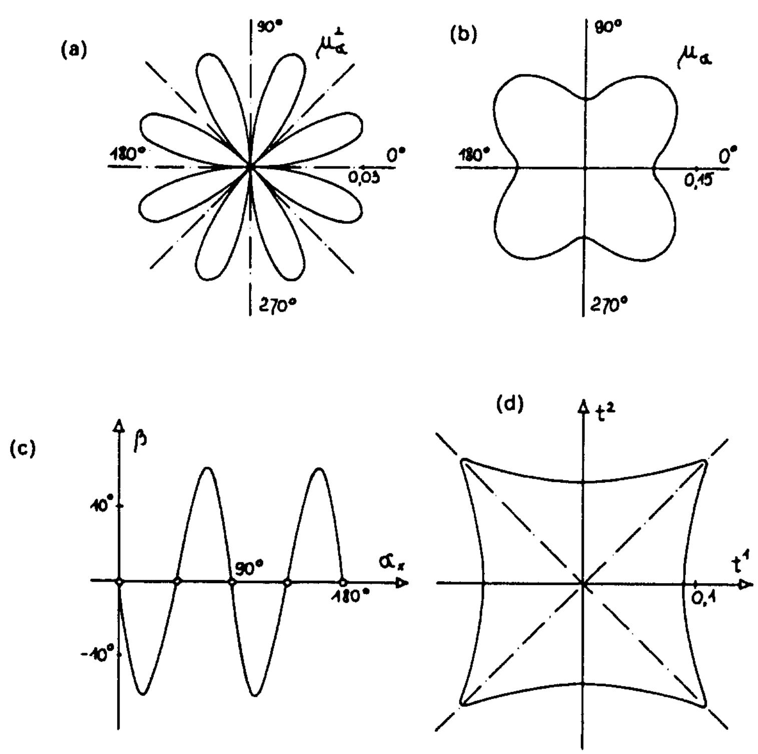
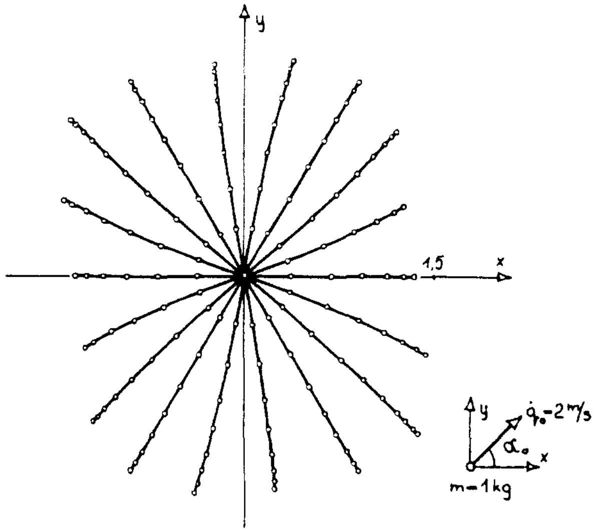
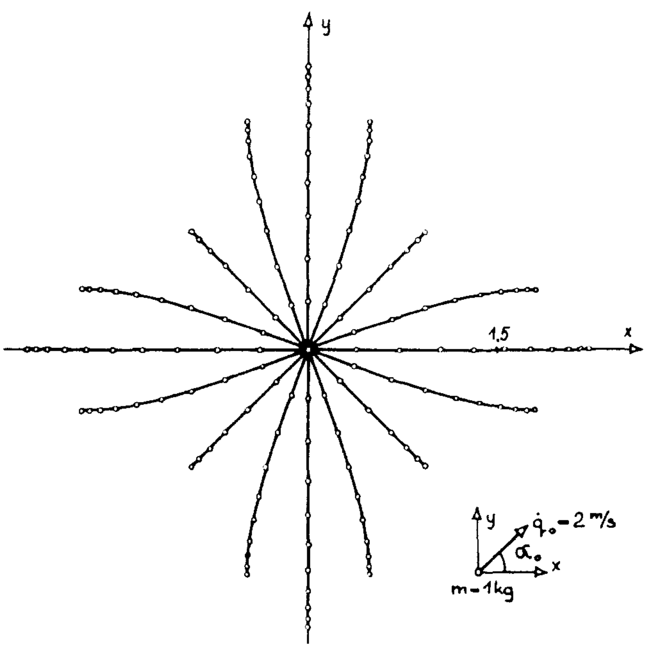


Fig. B3. Illustration of the tetragonal anisotropic friction: (a) friction coefficient $\mu_\alpha^\perp$; (b) friction coefficient μ_α; (c) inclination angle β; (d) friction force hodograph.



α_0	0°	20°	40°	60°	80°	100°	120°	140°	160°
	180°	200°	220°	240°	260°	280°	300°	320°	340°

Fig. B4. Motion of a material point in a plane with anisotropic friction with three principal directions of friction.



α_0	0°	22,5°	45°	67,5°	90°	112,5°	135°	157,5°
	180°	202,5°	225°	247,5°	270°	292,5°	315°	337,5°

Fig. B5. Motion of a material point in a plane with tetragonal anisotropic friction.